

PROGRESS IN NONCOMMUTATIVE FUNCTION THEORY

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*To our esteemed friend and teacher,
Richard V. Kadison,
on the happy occasion of his 85th birthday*

ABSTRACT. In this expository paper we describe the study of certain non-self-adjoint operator algebras, the Hardy algebras, and their representation theory. We view these algebras as algebras of (operator valued) functions on their spaces of representations. We will show that these spaces of representations can be parameterized as unit balls of certain W^* -correspondences and the functions can be viewed as Schur class operator functions on these balls. We will provide evidence to show that the elements in these (non commutative) Hardy algebras behave very much like bounded analytic functions and the study of these algebras should be viewed as noncommutative function theory.

1. INTRODUCTION

In this paper we shall introduce the tensor and Hardy operator algebras and discuss how to study them as algebras of operator valued functions on their representation spaces.

Tensor algebras associated with a bimodule over a ring have been studied extensively in a purely algebraic setting. This class of algebras has proved to be very important. In fact, every finite dimensional algebra is a quotient of a tensor algebra.

Looking for a similar class of operator algebras, we were led by the pioneering work of Pimsner [28] to study operator algebras associated with C^* -correspondences. A C^* -correspondence is, roughly, a bimodule over a C^* -algebra M that is also a (right) Hilbert C^* -module (see Section 2 below for more details).

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These operator algebras, which we call tensor algebras, are subalgebras of the C^* -algebras studied by Pimsner and are closely related to them. Both are generated (as a norm-closed algebra and as a C^* -algebra, respectively) by “shifts” on the Fock space of the correspondence. In fact, the Cuntz-Pimsner algebra associated to a given correspondence E can be shown to be the “minimal” C^* -algebra that is generated by the tensor algebra of E . We shall not need this here but details can be found in [21] and [14].

In this paper we shall take the C^* -algebra M to be a W^* -algebra and assume that E is a W^* -correspondence (details and definitions are in the next section). This allows us to take the ultra-weak closure of the tensor algebra. We call this ultra-weakly closed algebra the Hardy algebra associated with the correspondence. As we shall see below, the Hardy algebra that we get in the simplest case (where $M = \mathbb{C} = E$) is simply the classical Hardy algebra $H^\infty(\mathbb{D})$. The Hardy algebras associated with general W^* -correspondences are the main object of our study here.

When studying the representations of the tensor algebras, we realized that they can be parameterized by points in the closed unit balls of certain W^* -correspondences. This fact will be exploited when we view the elements of the tensor or the Hardy algebras as functions on the representation space.

Considering the elements of an algebra as functions on the set of its representations is not new, of course. It was done in a purely algebraic setting and in the setting of Banach or C^* -algebras. But, as we shall see, the fact that the representation space here can be viewed as a unit ball of a W^* -correspondence, will allow us to view these algebras as generalizations of algebras of holomorphic functions on the disc \mathbb{D} in \mathbb{C} .

In the next section we shall define the tensor and Hardy algebras and describe some of their basic properties. As we shall see, these algebras are generated by a copy of the W^* -algebra M and a copy of the correspondence E .

In Section 3 we study the representation theory of the tensor and Hardy algebras. We shall first discuss the (completely contractive) representations of the tensor algebras. For this, we fix a normal representation σ of M on a Hilbert space H and then show that all the representations of the algebra whose restriction to the copy of M is σ can be parameterized by the points of the closed unit ball of a certain W^* -correspondence (that we call the σ -dual of E and write E^σ for it).

In order to study the (completely contractive, ultra-weakly continuous) representations of the Hardy algebra, we have to find out what

representations of the tensor algebra can be extended to such representations of the Hardy algebra. This is done in Subsection 3.2. In this way, we identify the ultra-weakly continuous representations of the Hardy algebra $H^\infty(E)$ as a subset of the closed unit ball of E^σ that contains the open unit ball. We write $AC(E^\sigma)$ for this set.

Given a point $\eta \in AC(E^\sigma)$, the associated representation of the Hardy algebra will be denoted $\eta^* \times \sigma$ and, given an element $X \in H^\infty(E)$, we write

$$(1) \quad \widehat{X}(\eta^*) = (\eta^* \times \sigma)(X).$$

The reason for evaluating the function at η^* and not at η is technical and will be clarified later.

We, thus, obtain the transform $X \mapsto \widehat{X}$, where \widehat{X} is an operator valued function. We have already discussed the domain of these functions. In Section 4 we discuss the nature of these functions and we shall see that, up to a constant multiple, they form a natural generalization of the classical Schur class functions. We present two characterizations of these functions and call them Schur class operator functions.

In the last two sections we take a closer look at the transform $X \mapsto \widehat{X}$. In Section 5 we discuss the kernel of the transform and in the last section we note that we are really dealing with several transforms: for each normal representation σ of M we get a different transform and we discuss the relationships among them.

Along the way, we present several results that demonstrate our main point of view: These Hardy algebras form a useful analogue of the algebra of holomorphic functions on the disc \mathbb{D} and their study can be seen as noncommutative function theory.

2. INTRODUCING THE TENSOR AND THE HARDY ALGEBRAS

Before we introduce the algebras, we describe the setup. Throughout this paper, M will denote a fixed W^* -algebra. We do not preclude the possibility that M may be finite dimensional. Indeed, the situation when $M = \mathbb{C}^d$ can be very interesting (even for $d = 1$). However, we want to think of M abstractly, as a C^* -algebra that is a dual space, without regard to any Hilbert space on which M might be represented. The weak-* topology on a W^* -algebra or on any of its weak-* closed subspaces will be referred to as the *ultra-weak* topology.

To eliminate unnecessary technicalities, we shall always assume M is σ -finite in the sense that every family of mutually orthogonal projections in M is countable. Alternatively, to say M is σ -finite is to say that M has a faithful normal representation on a separable Hilbert

space. So, unless explicitly indicated otherwise, every Hilbert space we consider will be assumed to be separable.

In addition, E will denote a W^* -correspondence over M in the sense of [25]. For the definition, recall first that a (right) Hilbert C^* -module over M is a right module E over M that is also equipped with an M -valued inner product. More explicitly, we have a function $\langle \cdot, \cdot \rangle : E \times E \rightarrow M$ such that, for $\xi, \eta \in E$ and $a \in M$,

1. $\zeta \mapsto \langle \xi, \zeta \rangle$ is linear,
2. $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$
3. $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*$,
4. $\langle \xi, \xi \rangle \geq 0$, with $\langle \xi, \xi \rangle = 0$ only if $\xi = 0$, and
5. E is complete in the norm $\|\xi\| := \|\langle \xi, \xi \rangle\|^{1/2}$.

Such a C^* -module is said to be self-dual provided each (right) module map Φ from E into M is induced by a vector in E , i.e., there is an $\eta \in E$ such that $\Phi(\xi) = \langle \eta, \xi \rangle$, for all $\xi \in E$.

A self-dual Hilbert C^* -module E over a W^* -algebra M is said to be a W^* -module. Our basic reference for Hilbert C^* - and W^* -modules is [17]. It is shown in [17, Proposition 3.3.4] that when E is a self-dual Hilbert module over a W^* -algebra M , then E must be a dual space. In fact, it may be viewed as an ultra-weakly closed subspace of a W^* -algebra. Further, every continuous module map on E is adjointable [17, Corollary 3.3.2] and the algebra $\mathcal{L}(E)$ consisting of all continuous module maps on E is a W^* -algebra [17, Proposition 3.3.4].

Given a W^* -module E over M and a normal $*$ -representation σ of M on a Hilbert space H , one can define on the algebraic tensor product, $E \otimes H$, a (scalar valued) inner product that satisfies $\langle \xi \otimes h, \eta \otimes k \rangle = \langle h, \sigma(\langle \xi, \eta \rangle_E) k \rangle_H$. The completion of this inner-product space is a Hilbert space and we write $E \otimes_\sigma H$ for it. One can then define the induced representation σ^E of $\mathcal{L}(E)$ on $E \otimes_\sigma H$ by

$$(2) \quad \sigma^E(X)(\xi \otimes h) = X\xi \otimes h, \quad X \in \mathcal{L}(E), \xi \in E, h \in H.$$

We shall also write $X \otimes I_H$ for $\sigma^E(X)$.

Definition 2.1. *Let E be a W^* -module over the W^* -algebra M . We say that E is a W^* -correspondence over M if there is an ultra-weakly continuous $*$ -representation $\varphi : M \rightarrow \mathcal{L}(E)$ such that E becomes a bimodule over M where the left action of M is determined by φ_E (or simply φ), $a \cdot \xi = \varphi(a)\xi$.*

We shall assume that E is essential or non-degenerate as a left M -module. This is the same as assuming that φ is unital.

We also shall assume that our W^* -correspondences are countably generated as self-dual Hilbert modules over their coefficient algebras. This is equivalent to assuming that $\mathcal{L}(E)$ is σ -finite.

Example 2.2. (Basic Example) If $M = \mathbb{C}$, then a W^* -correspondence over M is simply a Hilbert space.

Example 2.3. Let $G = (G^0, G^1, r, s)$ be a directed graph. For simplicity we assume that G is finite. Thus both the set of vertices, G^0 , and the set of edges, G^1 , are finite; and $r, s : G^1 \rightarrow G^0$ are the range and source maps. We set $M = \ell^\infty(G^0)$ (so that M is simply \mathbb{C}^n , for some n , viewed as a W^* -algebra), and we set $E = \ell^\infty(G^1)$. Then we endow E with the structure of a W^* -correspondence via the formulas:

$$(\varphi(a)\xi b)(e) = a(r(e))\xi(e)b(s(e)) , \quad a, b \in M, \quad \xi \in E, \quad e \in G^1,$$

and

$$\langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \langle \xi(e), \eta(e) \rangle, \quad \xi, \eta \in E, \quad v \in G^0.$$

One can easily check that every W^* -correspondence over a finite dimensional commutative W^* -algebra is associated in this way with a finite directed graph.

Example 2.4. Let M be an arbitrary (σ -finite) W^* -algebra and let $\alpha : M \rightarrow M$ be a normal $*$ -endomorphism. Let $E = M$ (as a vector space) with right action given by multiplication, left action given by $\varphi = \alpha$ and inner product $\langle \xi, \eta \rangle := \xi^* \eta$. We denote this correspondence by ${}_\alpha M$. (If α is the identity, we write simply M for this correspondence).

Example 2.5. Let Φ be a normal, contractive, completely positive map on the W^* -algebra M . Write $E = M \otimes_\Phi M$. This is the W^* -correspondence obtained as the self-dual completion of the algebraic tensor product $M \otimes M$ with the inner product defined by $\langle a \otimes b, c \otimes d \rangle = b^* \Phi(a^* c) d$ and the bimodule structure defined by left and right multiplication: $\varphi(c)(a \otimes b) d = ca \otimes bd$. This correspondence was used by Popa [29], Mingo [19], Anantharam-Delarouche [1] and others to study the map Φ . It is referred to as the GNS correspondence of Φ . If Φ is an automorphism, $M \otimes_\Phi M$ is isomorphic to ${}_\Phi M$.

Along with E , we may form the (W^*) -tensor powers of E , $E^{\otimes n}$. They will be understood to be the self-dual completions of the C^* -tensor powers of E . Recall that the C^* -tensor product of two correspondences E and F over M is the completion of the algebraic (balanced) tensor product $E \otimes F$ with respect to the inner product

$$\langle \xi_1 \otimes \zeta_1, \xi_2 \otimes \zeta_2 \rangle = \langle \zeta_1, \varphi_F(\langle \xi_1, \xi_2 \rangle_E) \zeta_2 \rangle_F, \quad \xi_1, \xi_2 \in E, \quad \zeta_1, \zeta_2 \in F$$

and the bimodule structure is defined by

$$\varphi_{E \otimes F}(a)(\xi \otimes \zeta)b = \varphi_E(a)\xi \otimes \zeta b, \quad \xi \in E, \zeta \in F, a, b \in M.$$

Likewise, the Fock space over E , $\mathcal{F}(E)$, will be the self-dual completion of the Hilbert C^* -module direct sum of the $E^{\otimes n}$:

$$\mathcal{F}(E) = M \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \dots$$

We view $\mathcal{F}(E)$ as a W^* -correspondence over M , where the left and right actions of M are the obvious ones, i.e., the diagonal actions, and we shall write φ_∞ for the left diagonal action of M . Thus, for $\xi_1 \otimes \xi_2 \otimes \dots \otimes \xi_k \in E^{\otimes k}$ and $a \in M$,

$$\varphi_\infty(a)(\xi_1 \otimes \xi_2 \otimes \dots \otimes \xi_k) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \dots \otimes \xi_k.$$

For $\xi \in E$, we shall write T_ξ for the so-called *creation operator* on $\mathcal{F}(E)$ defined by the formula $T_\xi \eta = \xi \otimes \eta$, $\eta \in \mathcal{F}(E)$. It is easy to see that T_ξ is in $\mathcal{L}(\mathcal{F}(E))$ with norm $\|\xi\|$, and that T_ξ^* annihilates M , as a summand of $\mathcal{F}(E)$, while on elements of the form $\zeta \otimes \eta$, $\zeta \in E$, $\eta \in \mathcal{F}(E)$, it is given by the formula

$$T_\xi^*(\zeta \otimes \eta) := \varphi_\infty(\langle \xi, \zeta \rangle) \eta.$$

We are now ready to define the operator algebras.

Definition 2.6. *If E is a W^* -correspondence over a W^* -algebra M , then **the tensor algebra** of E , denoted $\mathcal{T}_+(E)$, is defined to be the norm-closed subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by $\varphi_\infty(M)$ and $\{T_\xi \mid \xi \in E\}$. The **Hardy algebra** of E , denoted $H^\infty(E)$, is defined to be the ultra-weak closure in $\mathcal{L}(\mathcal{F}(E))$ of $\mathcal{T}_+(E)$.*

Example 2.7. *If $M = E = \mathbb{C}$, the Fock correspondence is the Hilbert space ℓ^2 and, for $\xi = 1 \in \mathbb{C}$, T_1 is the unilateral shift. The tensor algebra in this case is the norm-closed algebra generated by the shift and can be identified with the disc algebra $A(\mathbb{D})$. The Hardy algebra is its w^* -closure and can be identified with $H^\infty(\mathbb{D})$.*

It will be useful to bear this example in mind as we proceed because our algebras, in general, can be viewed as noncommutative analogues of the disc and the (classical) Hardy algebras.

Example 2.8. *If $M = \mathbb{C}$ and $E = \mathbb{C}^d$, then the Fock correspondence is the Hilbert space $\ell^2(\mathbb{F}_d^+)$ where \mathbb{F}_d^+ is the free semigroup on d generators. Letting $\{e_i : 1 \leq i \leq d\}$ be the standard orthonormal basis of $E = \mathbb{C}^d$, we see that the tensor algebra is generated (as a norm-closed algebra) by the d shifts $\{T_{e_i} : 1 \leq i \leq d\}$ and the Hardy algebra is its w^* -closure. These algebras were studied extensively by Popescu (e.g. [30]), Davidson and Pitts (e.g. [7]) and others. Popescu denoted this tensor*

algebra \mathcal{A}_d (and called it the noncommutative disc algebra). The Hardy algebra was denoted F_d^∞ by Popescu and \mathcal{L}_d by Davidson and Pitts.

More examples are given in [25] and discussed in detail there.

An important tool used in the analysis of $\mathcal{T}_+(E)$ and $H^\infty(E)$ is the “spectral theory of the gauge automorphism group”. What we need is developed in detail in [25, Section 2]. Here we merely recall the essentials. The reader should keep in mind that its primary role is to allow us to handle in an analytic way the natural gradings that the Fock space and the Hardy algebra have. Let P_n denote the projection of $\mathcal{F}(E)$ onto $E^{\otimes n}$. Then $P_n \in \mathcal{L}(\mathcal{F}(E))$ and the series

$$W_t := \sum_{n=0}^{\infty} e^{int} P_n$$

converges in the ultra-weak topology on $\mathcal{L}(\mathcal{F}(E))$. The family $\{W_t\}_{t \in \mathbb{R}}$ is an ultra-weakly continuous, 2π -periodic unitary representation of \mathbb{R} in $\mathcal{L}(\mathcal{F}(E))$. Further, if $\{\gamma_t\}_{t \in \mathbb{R}}$ is defined by the formula $\gamma_t = \text{Ad}(W_t)$, then $\{\gamma_t\}_{t \in \mathbb{R}}$ is an ultra-weakly continuous group of $*$ -automorphisms of $\mathcal{L}(\mathcal{F}(E))$ that leaves invariant $\mathcal{T}_+(E)$ and $H^\infty(E)$. Indeed, the subalgebra of $H^\infty(E)$ fixed by $\{\gamma_t\}_{t \in \mathbb{R}}$ is $\varphi_\infty(M)$ and $\gamma_t(T_\xi) = e^{-it} T_\xi$, $\xi \in E$. Associated with $\{\gamma_t\}_{t \in \mathbb{R}}$ we have the “Fourier coefficient operators” $\{\Phi_j\}_{j \in \mathbb{Z}}$ on $\mathcal{L}(\mathcal{F}(E))$, which are defined by the formula

$$(3) \quad \Phi_j(a) := \frac{1}{2\pi} \int_0^{2\pi} e^{-int} \gamma_t(a) dt, \quad a \in \mathcal{L}(\mathcal{F}(E)),$$

where the integral converges in the ultra-weak topology. An alternate formula for Φ_j is

$$\Phi_j(a) = \sum_{k \in \mathbb{Z}} P_{k+j} a P_k.$$

Each Φ_j leaves $H^\infty(E)$ invariant and, in particular, $\Phi_j(T_{\xi_1} T_{\xi_2} \cdots T_{\xi_n}) = T_{\xi_1} T_{\xi_2} \cdots T_{\xi_n}$ if and only if $n = j$ and zero otherwise. Associated with the Φ_j are the “arithmetic mean operators” $\{\Sigma_k\}_{k \geq 1}$ that are defined by the formula

$$\Sigma_k(a) := \sum_{|j| < k} \left(1 - \frac{|j|}{k}\right) \Phi_j(a),$$

$a \in \mathcal{L}(\mathcal{F}(E))$. For $a \in \mathcal{L}(\mathcal{F}(E))$, $\lim_{k \rightarrow \infty} \Sigma_k(a) = a$, where the limit is taken in the ultra-weak topology.

Note that, for $X \in H^\infty(E)$ and $k \geq 1$, $\Phi_k(X) = T_{\xi_k}$ for some $\xi_k \in E^{\otimes k}$ and $\Phi_0(X) = \varphi_\infty(a)$ for some $a \in M$. We can write the

"Fourier expansion" of X

$$(4) \quad X \sim \Phi_0(X) + \Phi_1(X) + \Phi_2(X) + \cdots = \varphi_\infty(a) + T_{\xi_1} + T_{\xi_2} + \cdots .$$

3. THE REPRESENTATIONS OF THE TENSOR AND THE HARDY ALGEBRAS

We now turn to describe the representation theory of $\mathcal{T}_+(E)$ and $H^\infty(E)$. Details for what we describe are presented in Section 2 of [25] and in [27].

3.1. Representations of the tensor algebras. We start by discussing the representations of the tensor algebra $\mathcal{T}_+(E)$.

We shall consider only completely contractive representations and, in fact, only those completely contractive representations of $\mathcal{T}_+(E)$ with the property that $\rho \circ \varphi_\infty$ is an ultra-weakly continuous representation of M . This is not a significant restriction. In particular, it is not a restriction at all, if H is assumed to be separable, since every C^* -representation of a σ -finite W^* -algebra on a separable Hilbert space is automatically ultra-weakly continuous [37, Theorem V.5.1].

Note that, in the purely algebraic setting, where M is a ring and E is an M -bimodule, the representations of the (algebraic) tensor algebra are given by bimodule maps on E .

Here, suppose ρ is a completely contractive representation of $\mathcal{T}_+(E)$ on a Hilbert space H as above, then $\sigma := \rho \circ \varphi_\infty$ is a normal $*$ -representation of M on H and ρ defines a bimodule map T from E to $B(H)$ by the formula

$$T(\xi) := \rho(T_\xi).$$

To say that $T(\cdot)$ is a bimodule map means simply that $T(\varphi(a)\xi b) = \sigma(a)T(\xi)\sigma(b)$ for all $a, b \in M$ and for all $\xi \in E$. The assumption that ρ is completely contractive guarantees that T is completely contractive with respect to the unique operator space structure on E that arises from viewing E as a corner of its linking algebra.

Definition 3.1. *Let E be a W^* -correspondence over a W^* -algebra M . Then:*

- (1) *A completely contractive covariant representation of E on a Hilbert space H is a pair (T, σ) , where*
 - (a) *σ is a normal $*$ -representation of N in $B(H)$.*
 - (b) *T is a linear, completely contractive map from E to $B(H)$ that is continuous in the σ -topology of [4] on E and the ultraweak topology on $B(H)$.*

(c) T is a bimodule map in the sense that

$$T(\varphi(a)\xi b) = \sigma(a)T(\xi)\sigma(b), \quad \xi \in E, \quad a, b \in M.$$

(2) A completely contractive covariant representation (T, σ) of E in $B(H)$ is called *isometric in case*

$$(5) \quad T(\xi)^*T(\eta) = \sigma(\langle \xi, \eta \rangle)$$

for all $\xi, \eta \in E$.

The discussion above shows that every completely contractive representation ρ of $\mathcal{T}_+(E)$ on H gives rise to a completely contractive covariant representation of E on H . The following theorem shows that the converse also holds and it can be viewed as a generalized von Neumann inequality.

Theorem 3.2. *Let E be a W^* -correspondence over a von Neumann algebra M . To every completely contractive covariant representation, (T, σ) , of E there is a unique completely contractive representation ρ of the tensor algebra $\mathcal{T}_+(E)$ that satisfies*

$$\rho(T_\xi) = T(\xi) \quad \xi \in E$$

and

$$\rho(\varphi_\infty(a)) = \sigma(a) \quad a \in M.$$

The map $(T, \sigma) \mapsto \rho$ is a bijection between the set of all completely contractive covariant representations of E and all completely contractive (algebra) representations of $\mathcal{T}_+(E)$ whose restrictions to $\varphi_\infty(M)$ are continuous with respect to the ultraweak topology on $\mathcal{L}(\mathcal{F}(E))$.

Definition 3.3. *If (T, σ) is a completely contractive covariant representation of a W^* -correspondence E over a von Neumann algebra M , we call the representation ρ of $\mathcal{T}_+(E)$ described in Theorem 3.2 the integrated form of (T, σ) and write $\rho = T \times \sigma$.*

As we showed in [21, Lemmas 3.4–3.6], and in [25], if a completely contractive covariant representation, (T, σ) , of E in $B(H)$ is given, then it determines a contraction $\tilde{T} : E \otimes_\sigma H \rightarrow H$ defined by the formula $\tilde{T}(\eta \otimes h) := T(\eta)h$, $\eta \otimes h \in E \otimes_\sigma H$. The operator \tilde{T} intertwines the representation σ on H and the induced representation $\sigma^E := \varphi(\cdot) \otimes I_H$ of M on $E \otimes_\sigma H$; i.e.

$$(6) \quad \tilde{T}(\varphi(\cdot) \otimes I) = \sigma(\cdot)\tilde{T}.$$

In fact we have the following lemma from [25, Lemma 2.16],.

Lemma 3.4. *The map $(T, \sigma) \rightarrow \tilde{T}$ is a bijection between all completely contractive covariant representations (T, σ) of E on the Hilbert space H and contractive operators $\tilde{T} : E \otimes_\sigma H \rightarrow H$ that satisfy equation (6). Given such a \tilde{T} satisfying this equation, T , defined by the formula $T(\xi)h := \tilde{T}(\xi \otimes h)$, together with σ is a completely contractive covariant representation of E on H . Further, (T, σ) is isometric if and only if \tilde{T} is an isometry.*

Associated with (T, σ) we also have maps $\tilde{T}_n : E^{\otimes n} \otimes H \rightarrow H$ defined by $\tilde{T}_n(\xi_1 \otimes \xi_2 \cdots \otimes \xi_n \otimes h) = T(\xi_1)T(\xi_2) \cdots T(\xi_n)h$.

Now fix a normal representation σ of M on a Hilbert space H . The discussion above shows that the set of all the completely contractive representations ρ of the tensor algebra $\mathcal{T}_+(E)$ that satisfy $\rho \circ \varphi_\infty = \sigma$ (roughly speaking, ρ , restricted to M is σ) can be parameterized by the contractions $\tilde{T} \in B(E \otimes_\sigma H, H)$ that satisfy the intertwining relation (6). This is, of course, the same as saying that this set of representations are parameterized by the adjoints \tilde{T}^* . The reason that we prefer to consider the adjoints is that the set of all the maps \tilde{T}^* satisfying relation (6) can be given the structure of a W^* -correspondence as the following proposition shows.

Proposition 3.5. *Let E be a W^* -correspondence over the W^* -algebra M and let σ be a normal representation of M on the Hilbert space H . Write E^σ for the space of all bounded maps $\eta : H \rightarrow E \otimes_\sigma H$ that satisfy*

$$(7) \quad \eta\sigma(a) = (\varphi_\infty(a) \otimes I_H)\eta, \quad a \in M.$$

With respect to the action of $\sigma(M)'$ and the $\sigma(M)'$ -valued inner product defined as follows, E^σ becomes a W^ -correspondence over $\sigma(M)'$: For $X, Y \in \sigma(M)'$, and $T \in E^\sigma$, $X \cdot T \cdot Y := (I \otimes X)TY$, and for $T, S \in E^\sigma$, $\langle T, S \rangle := T^*S$.*

Definition 3.6. *The W^* -correspondence of Proposition 3.5 will be called the σ -dual of E .*

From equation (6) we see that \tilde{T}^* lies in the space we have denoted E^σ . So, if we write $\mathbb{D}(E^\sigma)$ for the open unit ball in E^σ and $\overline{\mathbb{D}(E^\sigma)}$ for its norm closure, then all the completely contractive representations ρ of $\mathcal{T}_+(E)$ such that $\rho \circ \varphi_\infty = \sigma$ are parametrized bijectively by $\overline{\mathbb{D}(E^{\sigma*})} = \overline{\mathbb{D}(E^\sigma)^*} = \overline{\mathbb{D}(E^\sigma)}^*$.

Example 3.7. *In the special case when (E, M) is $(\mathbb{C}^d, \mathbb{C})$, a representation σ of \mathbb{C} on a Hilbert space H is quite simple; it does the only thing it can: $\sigma(c)h = ch$, $h \in H$, and $c \in \mathbb{C}$. In this setting, $E \otimes_\sigma H$*

is just the direct sum of d copies of H and \tilde{T} is simply a d -tuple of operators (T_1, T_2, \dots, T_d) such that $\|\sum_i T_i T_i^*\| \leq 1$, i.e. \tilde{T} is a row contraction. The map T , then, is given by the formula $T(\xi) = \sum \xi_i T_i$, where $\xi = (\xi_1, \xi_2, \dots, \xi_d)^\top \in \mathbb{C}^d$. The space E^σ is column space over $B(H)$, $\mathbf{C}_d(B(H))$, and $\mathbb{D}(E^\sigma)$ is simply the unit ball in $\mathbf{C}_d(B(H))$.

It follows from Pimsner's analysis that (T, σ) is isometric if and only if $T \times \sigma$ is the restriction to $\mathcal{T}_+(E)$ of a C^* -representation of the C^* -subalgebra $\mathcal{T}(E)$ of $\mathcal{L}(\mathcal{F}(E))$ generated by $\mathcal{T}_+(E)$. This C^* -algebra is called the *Toeplitz algebra* of E .

A special kind of isometric covariant representations that will play an important role here are constructed as follows. Let $\pi_0 : M \rightarrow B(H_0)$ be a normal representation of M on the Hilbert space H_0 , and let $H = \mathcal{F}(E) \otimes_{\pi_0} H_0$. Set $\sigma := \pi^{\mathcal{F}(E)} \circ \varphi_\infty = \varphi_\infty(\cdot) \otimes I_{H_0}$, and define $S : E \rightarrow B(H)$ by the formula $S(\xi) = T_\xi \otimes I_{H_0}$, $\xi \in E$. Then it is immediate that (S, σ) is an isometric covariant representation and we say that it is *induced by* π_0 . We also will say $S \times \sigma$ is induced by π_0 . In fact,

$$(8) \quad S \times \sigma = \pi_0^{\mathcal{F}(E)}|_{\mathcal{T}_+(E)}.$$

In a sense that will become clear, an induced representations should be viewed as a generalization of a unilateral shift where the representation π_0 plays the role of the multiplicity of the shift.

An induced isometric covariant representation has the property that $\widetilde{S_n S_n^*} \rightarrow 0$ strongly as $n \rightarrow \infty$ because $\widetilde{S_n S_n^*}$ is the projection onto $\sum_{k \geq n} E^{\otimes k} \otimes_{\pi_0} H_0$. In general, an isometric covariant representation (S, σ) and its integrated form are called *pure* if $\widetilde{S_n S_n^*} \rightarrow 0$ strongly as $n \rightarrow \infty$.

Corollary 2.10 of [22] shows that every pure isometric covariant representation of (E, M) is unitarily equivalent to an isometric covariant representation that is induced by a normal representation of M . We therefore will usually say simply that a pure isometric covariant representation *is* induced. In Theorem 2.9 of [22] we proved a generalization of the Wold decomposition theorem that asserts that every isometric covariant representation of (E, M) decomposes as the direct sum of an induced isometric covariant representation of (E, M) and an isometric representation of (E, M) that is both isometric and fully coisometric.

We will need an analogue of a unilateral shift of infinite multiplicity. For that, we shall fix, once and for all, a representation (S_0, σ_0) that is induced by a faithful normal representation π of M that has *infinite multiplicity*. That is, (S_0, σ_0) acts on a Hilbert space of the form $\mathcal{F}(E) \otimes_\pi K_0$, where $\pi : M \rightarrow B(K_0)$ is an infinite ampliation

of a faithful normal representation of M . Then $\sigma_0 := \pi^{\mathcal{F}(E)} \circ \varphi_\infty$, while $S_0(\xi) := T_\xi \otimes I_{K_0}$, $\xi \in E$. The following proposition shows the uniqueness and the special role of this representation.

Proposition 3.8. *The representation (S_0, σ_0) is unique up to unitary equivalence and every induced isometric covariant representation of (E, M) is unitarily equivalent (in a natural way) to a restriction of (S_0, σ_0) to a subspace of the form $\mathcal{F}(E) \otimes_\pi \mathfrak{K}$, where \mathfrak{K} is a subspace of K_0 that reduces π .*

Definition 3.9. *We shall refer to (S_0, σ_0) as the universal induced covariant representation of (E, M) .*

By Proposition 3.8, (S_0, σ_0) does not really depend on the choice of representation π used to define it. It will serve the purpose in our theory that the unilateral shift of infinite multiplicity serves in the structure theory of single operators on Hilbert space.

A key tool in our theory is the following result that we proved as [25, Theorem 2.8].

Theorem 3.10. *Let (T, σ) be a completely contractive covariant representation of (E, M) on a Hilbert space H . Then there is an isometric covariant representation (V, τ) of (E, M) acting on a Hilbert space K containing H such that if P denotes the projection of K onto H , then*

- (1) *P commutes with $\tau(M)$ and $\tau(a)P = \sigma(a)P$, $a \in M$, and*
- (2) *for all $\eta \in E$, $V(\eta)^*$ leaves H invariant and $PV(\eta)P = T(\eta)P$.*

*The representation (V, τ) may be chosen so that the smallest subspace of K that contains H and is invariant under both $\tau(M)$ and $V(E)$, is all of K . When this is done, (V, τ) is unique up to unitary equivalence and is called **the minimal isometric dilation** of (T, σ) .*

Note that, in the notation of the theorem, we have

$$(9) \quad T \times \sigma = P(V \times \tau)P.$$

Thus, the representation $T \times \sigma$ is a compression, onto a coinvariant subspace, of the representation $V \times \tau$.

Another result that will be important when studying the representations of the tensor and the Hardy algebras is our version of the commutant lifting theorem. This theorem was proved in [21] and can be stated as follows (see [27, Theorems 2.6 and 2.7]).

Theorem 3.11. *For $i = 1, 2$, let (T_i, σ_i) be a completely contractive covariant representation of (E, M) on a Hilbert space H_i , let (V_i, τ_i) be the minimal isometric dilation of (T_i, σ_i) acting on the space K_i , and let P_i be the orthogonal projection of K_i onto H_i . Then, given an*

operator $X \in B(H_1, H_2)$ that intertwines the representations $T_1 \times \sigma_1$ and $T_2 \times \sigma_2$, there is an operator $Y \in B(K_1, K_2)$ such that

- (1) Y intertwines the representations $V_1 \times \tau_1$ and $V_2 \times \tau_2$,
- (2) $X = P_2 Y P_1$,
- (3) $Y H_1^\perp \subseteq H_2^\perp$ and
- (4) $\|Y\| = \|X\|$.

We end this section with a discussion of the representations of the tensor algebras associated with directed graphs (see Example 2.3).

Example 3.12. Let G and E as described in Example 2.3. Write $E(G)$ for E . The algebra $H^\infty(E)$ in this case will be written $H^\infty(G)$. In the literature, $H^\infty(G)$ is sometimes denoted \mathcal{L}_G . It is the ultraweak closure of the tensor algebra $\mathcal{T}_+(E(G))$ acting on the Fock space $\mathcal{F}(E(G))$. For $e \in G^1$, let δ_e be the δ -function at e , i.e., $\delta_e(e') = 1$ if $e = e'$ and is zero otherwise. Then T_{δ_e} is a partial isometry that we denote by S_e . Also, for $v \in G^0$, P_v is defined to be $\varphi_\infty(\delta_v)$. Then each P_v is a projection and it is an easy matter to see that the families $\{S_e : e \in G^1\}$ and $\{P_v : v \in G^0\}$ form a Cuntz-Toeplitz family in the sense that the following conditions are satisfied:

- (i) $P_v P_u = 0$ if $u \neq v$,
- (ii) $S_e^* S_f = 0$ if $e \neq f$
- (iii) $S_e^* S_e = P_{s(e)}$ and
- (iv) $\sum_{r(e)=v} S_e S_e^* \leq P_v$ for all $v \in G^0$.

The algebra $\mathcal{T}_+(E(G))$ was first defined and studied in [20], providing examples of the theory developed in [21]. It was called a quiver algebra there because in pure algebra, directed graphs are called quivers. The properties of quiver algebras were further developed in [22]. In [15], the focus was on $H^\infty(G)$ and the authors called this algebra a free semigroupoid algebras. Both algebras are often represented as algebras of operators on $l_2(G^*)$ (where G^* is the set of all finite paths in G), and it will be helpful to understand how this is done, from the perspective of this note. Let H_0 be a Hilbert space whose dimension equals the number of vertices, let $\{e_v \mid v \in G^0\}$ be a fixed orthonormal basis for H_0 and let π_0 be the diagonal representation of $M = \ell^\infty(G^0)$ on H_0 . Then $l_2(G^*)$ is isomorphic to $\mathcal{F}(E(G)) \otimes_{\pi_0} H_0$ where the isomorphism maps an element ξ_α of the standard orthonormal basis of $l_2(G^*)$ to $\delta_\alpha \otimes e_{s(e_k)}$ (where, for a finite path $\alpha = e_1 \cdots e_k$, $\delta_\alpha = \delta_{e_1} \otimes \cdots \otimes \delta_{e_k} \in E^{\otimes k}$). The partial isometries S_e can then be viewed as the shift operators $S_e \xi_\alpha = \xi_{e\alpha}$. Thus, the representations of $\mathcal{T}_+(E(G))$ and $H^\infty(G)$ on $l_2(G^*)$ are just the representations induced by π_0 .

Quite generally, a completely contractive covariant representation of $E(G)$ on a Hilbert space H is given by a representation σ of $M = \ell^\infty(G^0)$ on H and by a contractive map $\tilde{T} : E \otimes_\sigma H \rightarrow H$ satisfying equation (6). The representation σ is given by the projections $Q_v = \sigma(\delta_v)$ whose sum is I . Also, from \tilde{T} we may define maps $T(e) \in B(H)$ by the equation $T(e)h = \tilde{T}(\delta_e \otimes h)$ and it is easy to check that $\tilde{T}\tilde{T}^* = \sum_e T(e)T(e)^*$ and $T(e) = Q_{r(e)}T(e)Q_{s(e)}$. Thus to every completely contractive representation of the quiver algebra $\mathcal{T}_+(E(G))$ we associate a family $\{T(e) | e \in G^1\}$ of maps on H that satisfy $\sum_e T(e)T(e)^* \leq I$ and $T(e) = Q_{r(e)}T(e)Q_{s(e)}$. Conversely, every such family defines a representation, written $T \times \sigma$ (or $\tilde{T} \times \sigma$), satisfying $(T \times \sigma)(S_e) = T(e)$ and $(T \times \sigma)(P_v) = Q_v$.

Now we fix σ to be π_0 and write H in place of H_0 . So that, in this case, each projection Q_v is one dimensional (with range equal to $\mathbb{C}e_v$). Then obviously $\sigma(M)' = \sigma(M)$. To describe the σ -dual of E , write G^{-1} for the directed graph obtained from G by reversing all arrows, so that $s(e^{-1}) = r(e)$ and $r(e^{-1}) = s(e)$. Sometimes G^{-1} is denoted G^{op} and is called the opposite graph. Note that the Hilbert space $E \otimes_\sigma H_0$ is spanned by the orthonormal basis $\{\delta_e \otimes e_{s(e)}\}$. Fix $\eta \in E^\sigma$ and note that its covariance property implies that, for every $e \in G^1$, $\eta^*(\delta_e \otimes e_{s(e)}) = \eta^*(\delta_{r(e)}\delta_e \otimes e_{s(e)}) = Q_{r(e)}\eta^*(\delta_e \otimes e_{s(e)}) = \overline{\eta(e^{-1})}e_{r(e)}$ for some $\eta(e^{-1}) \in \mathbb{C}$. The reason for the "strange" way of writing that scalar is that now we can view η as an element of $E(G^{-1})$ and the correspondence structure on E^σ , as described in Proposition 3.5, fits the correspondence structure of $E(G^{-1})$. Consequently, we can identify the two and write

$$E^\sigma = E(G^{-1}).$$

(See Example 4.3 in [25] for a description of the structure of the dual correspondence for more general representations σ). It will also be convenient to write η matrixially with respect to the orthonormal bases $\{\delta_v \mid v \in G^0\}$ of H_0 and $\{\delta_e \otimes e_{s(e)}\}_{e \in G^1}$ of $E \otimes H_0$ as

$$(10) \quad (\eta)_{e,r(e)} = \eta(e^{-1}).$$

3.2. Representations of the Hardy algebras. We now turn to study the ultra-weakly continuous, completely contractive representations of the Hardy algebra $H^\infty(E)$. Given such a representation ρ , its restriction to the tensor algebra $\mathcal{T}_+(E)$ is a completely contractive representation of this algebra. Thus it is of the form $T \times \sigma$ for some $\tilde{T}^* \in E^\sigma$ (where $\sigma = \rho \circ \varphi_\infty$). Therefore, the problem we face is to decide when the integrated form, $T \times \sigma$, of a completely contractive covariant representation (T, σ) extends from $\mathcal{T}_+(E)$ to $H^\infty(E)$. This

problem arises already in the simplest situation, vis. when $M = \mathbb{C} = E$. In this setting, T is given by a single contraction operator on a Hilbert space, $\mathcal{T}_+(E)$ “is” the disc algebra $A(\mathbb{D})$ and $H^\infty(E)$ “is” the space $H^\infty(\mathbb{D})$ of bounded analytic functions on the disc. In this case it is known that the representation $T \times \sigma$ extends from the disc algebra to $H^\infty(\mathbb{D})$ precisely when there is no singular part to the spectral measure of the minimal unitary dilation of T . In our general context, one may be able to identify an analogue of a unitary dilation but it is rarely unique ([21]). Also, it doesn’t seem to have any analogue for a spectral measure. Thus we will need to use different tools.

One class of representations of the tensor algebra that extend to ultra-weakly continuous representations of $H^\infty(E)$ we have already met. These are the induced representations. In the notation of (8), the ultra-weakly continuous extension of $\pi_0^{\mathcal{F}(E)}|_{\mathcal{T}_+(E)}$ is $\pi_0^{\mathcal{F}(E)}|_{H^\infty(E)}$.

It was proved in [25, Theorem 2.13] that, if $\|\tilde{T}\| < 1$, then the minimal isometric dilation (V, τ) of (T, σ) (as in Theorem 3.10) is an induced representation. Thus $V \times \tau$ extends to an ultra-weakly continuous representation of $H^\infty(E)$. Since $T \times \sigma$ is a compression of $V \times \tau$, we have the following.

Lemma 3.13. [25, Corollary 2.14] *If $\|\tilde{T}\| < 1$ then $T \times \sigma$ extends to a ultraweakly continuous representation of $H^\infty(E)$.*

If $T \times \sigma$ is a representation of the tensor algebra on the space H that extends to a ultra-weakly continuous representation of $H^\infty(E)$ then, for every $x \in H$, the linear functional $f = \omega_x \circ (T \times \sigma)$ extends to a ultra-weakly continuous functional on $H^\infty(E)$. Given an arbitrary representation $T \times \sigma$, one can still consider the set of all vectors $x \in H$ with this property. For the case where $M = \mathbb{C}$, this was done in [9] and the following definition is a direct extension of their definition.

Definition 3.14. *Given a c.c. covariant representation (T, σ) on H , we say that $x \in H$ is **absolutely continuous** if the functional $\omega_x \circ (T \times \sigma)$, on $\mathcal{T}_+(E)$, extends to a ultraweakly continuous functional on $H^\infty(E)$ and we write $\mathcal{V}_{ac}(T, \sigma)$ for the set of all the absolutely continuous vectors for (T, σ) .*

It turns out that the set of absolutely continuous vectors can be studied by considering the ranges of certain intertwiners.

Definition 3.15. *Let (S_0, σ_0) be the universal induced covariant representation (see Definition 3.9). For a given $\eta \in \overline{\mathbb{D}(E^\sigma)}$ (corresponding to the representation (T, σ) on H) write $\mathcal{I}(S_0, \eta^*)$ (or $\mathcal{I}(S_0, \tilde{T})$) for the*

space of intertwiners: $\mathcal{I}(S_0, \eta^*) =$

$$\{C : H_0 \rightarrow H : CS_0(\xi) = T(\xi)C, C\sigma_0(a) = \sigma(a)C, \xi \in E, a \in M\}.$$

The role of these intertwiners for studying the ultraweakly continuous representations of $H^\infty(E)$ is seen in [10, Corollaries 5.4 and 5.5] (in the case $M = \mathbb{C}$) and in [25, Lemma 7.12] (in the general case). An immediate corollary of the latter lemma is that, when (T, σ) is an isometric representation, for every $C \in \mathcal{I}(S_0, \eta^*)$, the range $\text{Ran}(C)$ of C is contained in $\mathcal{V}_{ac}(T, \sigma)$. (Here $\eta = \tilde{T}^*$.) Generalizing some arguments of [9, Theorem 1.6], we prove in [27, Proposition 3.5] that the converse also holds. Thus we have the following.

Theorem 3.16. *If (T, σ) is an isometric covariant representation then*

$$\mathcal{V}_{ac}(T, \sigma) = \bigcup \{\text{Ran}(C) : C \in \mathcal{I}(S_0, \tilde{T})\}.$$

It follows that $\mathcal{V}_{ac}(T, \sigma)$ is a closed, $\sigma(M)$ -invariant subspace.

In order to analyze the set of all absolutely continuous vectors for a general completely contractive (not necessarily isometric) representation, we use the fact that every such representation (T, σ) on H has a minimal isometric dilation to an isometric representation (V, τ) on a larger space K (Theorem 3.10). This dilation is constructed explicitly in [21] and it is evident from the construction that the restriction of (V, τ) to $K \ominus H$ is an induced representation. It then follows that $K \ominus H \subseteq \mathcal{V}_{ac}(V, \tau)$. Another tool used in the proof of the following theorem, which shows the close relationship between $\mathcal{V}_{ac}(V, \tau)$ and $\mathcal{V}_{ac}(T, \sigma)$, is the commutant lifting theorem (Theorem 3.11). This theorem is applied to show that every operator C in $\mathcal{I}(S_0, \tilde{T})$ can be written as $P_H X$ for some $X \in \mathcal{I}(S_0, \tilde{V})$ (where P_H is the projection onto H).

Theorem 3.17. *Let (T, σ) be a completely contractive covariant representation of (E, M) on the Hilbert space H , let (V, ρ) be the minimal isometric dilation of (T, σ) acting on a Hilbert space K containing H , and let P denote the projection of K onto H . Then $K \ominus H$ is contained in $\mathcal{V}_{ac}(V, \rho)$ and the following sets are equal.*

- (1) $\mathcal{V}_{ac}(T, \sigma)$.
- (2) $H \cap \mathcal{V}_{ac}(V, \rho)$.
- (3) $P\mathcal{V}_{ac}(V, \rho)$.
- (4) $\bigcup \{\text{Ran}(C) \mid C \in \mathcal{I}(S_0, \tilde{T})\}$.

In particular, $\mathcal{V}_{ac}(T, \sigma) = H$ if and only if $\mathcal{V}_{ac}(V, \rho) = K$.

Definition 3.18. *Given a c.c. covariant representation (T, σ) , one defines the completely positive map associated with it, $\Phi_T :$*

$\sigma(M)' \rightarrow \sigma(M)'$ by

$$\Phi_T(b) = \eta^*(I_E \otimes b)\eta = \tilde{T}(I_E \otimes b)\tilde{T}^*$$

where $\eta = \tilde{T}^*$.

Example 3.19. If $M = \mathbb{C}$ and $E = \mathbb{C}^d$, every completely contractive representation is given by a row contraction $\tilde{T} = (T_1, \dots, T_d)$. In this case

$$\Phi_T(b) = \sum T_i b T_i^*.$$

Note that the map Φ_T is a completely positive, contractive, normal map on the von Neumann algebra $\sigma(M)'$. In fact, **every** completely positive, contractive, normal map on a von Neumann algebra N is of this form. (See [23, Corollary 2.23] for details).

Often properties of the representation (T, σ) can be expressed in terms of the associated map Φ_T . For example, the map is multiplicative (that is, a $*$ -endomorphism) if and only if the representation is isometric. (This is a rough statement. For the precise one see [23, Proposition 2.21]). Another example is the curvature associated with a representation which was shown in [24] to be an artifact of the associated map.

Here, too, we find that there is a close relationship between the intertwiners in $\mathcal{I}(S_0, \eta^*)$ and pure superharmonic elements (to be defined below) of the associated map.

Lemma 3.20. If $C \in \mathcal{I}(S_0, \eta^*)$, then $Q = CC^*$ lies in $\sigma(M)'$ and satisfies

- (i) $Q \geq 0$ and $\Phi_T(Q) \leq Q$, and
- (ii) $\Phi_T^n(Q) \rightarrow 0$ ultra weakly.

The proof of (i) follows from the fact that $CS_0(\xi) = T(\xi)C$ for $\xi \in E$ and, thus, $C\tilde{S}_0 = \tilde{T}(I_E \otimes C)$ and $\Phi_T(CC^*) = \tilde{T}(I_E \otimes CC^*)\tilde{T}^* = C\tilde{S}_0\tilde{S}_0^*C^* \leq CC^*$. The proof of (ii) follows similarly from the fact that S_0 is pure (that is, $\tilde{S}_{0n}\tilde{S}_{0n}^* \rightarrow 0$ strongly as $n \rightarrow \infty$).

Definition 3.21. An element $Q \in \sigma(M)'$ satisfying (i) of the lemma will be said to be superharmonic for Φ_T . If it also satisfied (ii), it will be said to be pure superharmonic.

Thus, for every $C \in \mathcal{I}(S_0, \eta^*)$, $Q = CC^*$ is pure superharmonic. The converse also holds. Given an element $Q \in \sigma(M)'$ that is pure superharmonic for Φ_T , define $r \in \sigma(M)'$ to be the positive square root of $Q - \Phi_T(Q)$. Then $\sum_{n \geq 0} \Phi_T^n(r^2) = Q$ (in the strong operator topology).

Let \mathcal{R} be the closure of the range of r and $\sigma_{\mathcal{R}} := \sigma|_{\mathcal{R}}$. Since π is a normal, faithful representation of M with infinite multiplicity and $\sigma_{\mathcal{R}}$ is a normal representation of M , there exists an isometry $v : \mathcal{R} \rightarrow K_0$ that intertwines $\sigma_{\mathcal{R}}$ and π . Then set

$$C^* = (I_{\mathcal{F}(E)} \otimes v) \sum_{n \geq 0} (I_{E^{\otimes n}} \otimes r) \tilde{T}_n^*$$

where $\tilde{T}_n = \tilde{T}(I_E \otimes \tilde{T}) \cdots (I_{E^{\otimes(n-1)}} \otimes \tilde{T}) : E^{\otimes n} \otimes_{\sigma} H \rightarrow H$.

It is then straightforward to show that $C \in \mathcal{I}(S_0, \tilde{T})$ and $CC^* = Q$. Some of the arguments above can be found, for $M = \mathbb{C}$, in [10] and in [32]. We conclude the following.

Corollary 3.22.

$$\begin{aligned} \mathcal{V}_{ac}(T, \sigma) &= \bigcup \{ \text{Ran}(C) : C \in \mathcal{I}(S_0, \tilde{T}) \} = \\ &= \bigvee \{ \text{Ran}(Q) : Q \text{ is a pure superharmonic operator for } \Phi_T \}. \end{aligned}$$

We can now state the following result which gives a complete description of the representations of $H^{\infty}(E)$ (see [27, Theorem 4.1]).

Theorem 3.23. *Let $T \times \sigma$ be a c.c. representation of $\mathcal{T}_+(E)$ on H and write $\eta = \tilde{T}^*$ for the element of $\overline{\mathbb{D}(E^{\sigma})}$ associated with it. Then the following are equivalent.*

- (1) *The representation $T \times \sigma$ extends to a completely contractive ultra weakly continuous representation of $H^{\infty}(E)$.*
- (2) $\mathcal{V}_{ac}(T, \sigma) = H$
- (3) $H = \bigvee \{ \text{Ran}(C) : C \in \mathcal{I}(S_0, \eta^*) \}$.
- (4) $H = \bigvee \{ \text{Ran}(Q) : Q \text{ is pure superharmonic for } \Phi_T \}$

Note that the equivalence of (1) and (2) of the theorem means that this "extension" problem can be studied "locally".

Theorem 3.23 describes the representations ρ of $H^{\infty}(E)$ that satisfy $\rho \circ \varphi_{\infty} = \sigma$. The set of the points $\eta \in \overline{\mathbb{D}(E^{\sigma})}$ that correspond to these representations will be denoted $AC(E^{\sigma})$. These sets (for all σ 's) parameterize the representations of $H^{\infty}(E)$. We have

$$\mathbb{D}(E^{\sigma}) \subseteq AC(E^{\sigma}) \subseteq \overline{\mathbb{D}(E^{\sigma})}.$$

As corollaries of the analysis above, we get the following.

Theorem 3.24. [27, Theorem 5.6] *If $\sigma(M)'$ is finite dimensional, then $\rho := T \times \sigma$ extends to an ultra-weakly continuous representation of $H^{\infty}(E)$ on H if and only if (T, σ) is completely non coisometric; that is, there is no $\rho(\mathcal{T}_+(E))^*$ -invariant subspace of H on which \tilde{T}^* is an isometry.*

Theorem 3.25. [27, Theorem 5.3] *If $\sigma(M)'$ has a non zero normal periodic state ω for Φ_T (that is, $\omega \circ \Phi_T^k = \omega$ for some $k \geq 1$) then $T \times \sigma$ does not extend to an ultra-weakly continuous representation of $H^\infty(E)$.*

When (T, σ) is an isometric representation, the space $\mathcal{V}_{ac}(T, \sigma)$ contains all the wandering vectors of $T \times \sigma$ where $h \in H$ is said to be wandering if, for every $n \neq m$, the subspaces $\tilde{T}_n(E^{\otimes n} \otimes [\sigma(M)h])$ and $\tilde{T}_m(E^{\otimes m} \otimes [\sigma(M)h])$ are orthogonal. This, and other results concerning wandering vectors, was proved in [27] generalizing similar results in [9] (who proved it for the case $M = \mathbb{C}$).

4. THE FUNCTIONS DEFINED BY ELEMENTS OF $H^\infty(E)$

As we stated in the introduction, we view the elements of $H^\infty(E)$ as functions defined on the space of the representations of $H^\infty(E)$.

As seen in the previous section, the space of all representations can be parameterized by $\cup AC(E^\sigma)^*$ (where the union runs over all normal representations σ of M). In the discussion below, we shall fix σ .

Now, a given $X \in H^\infty(E)$ will be viewed as a function \hat{X} , defined on $AC(E^\sigma)^*$ by the equation

$$(11) \quad \hat{X}(\eta^*) = (\eta^* \times \sigma)(X), \quad \eta \in AC(E^\sigma).$$

Often it will be more convenient to restrict the function \hat{X} to the open unit ball $\mathbb{D}(E^\sigma)^*$. This restriction will also be denoted \hat{X} .

The primary objective in this section is to understand the range of the transform

$$X \mapsto \hat{X}$$

from $H^\infty(E)$ to the set of all $B(H)$ -valued functions on $AC(E^\sigma)^*$ or on $\mathbb{D}(E^\sigma)^*$.

Before we do this we note that this map depends on σ and that, for a given σ , it may have a non zero kernel; that is, there may be some $X \in H^\infty(E)$ such that $\hat{X} = 0$. However, it was shown in [26, Lemma 5.7] that we can always choose an appropriate σ so that this transform is injective. We shall have more to say about the kernel of the transform in Section 5.

Example 4.1. *Suppose $M = E = \mathbb{C}$ and σ the representation of \mathbb{C} on some Hilbert space H . Then it is easy to check that E^σ is isomorphic to $B(H)$. Fix an $X \in H^\infty(E)$. As we mentioned above, this Hardy algebra is the classical $H^\infty(\mathbb{D})$ and we can identify X with a function $f \in H^\infty(\mathbb{T})$. Given $S \in \mathbb{D}(E^\sigma) = B(H)$, it is not hard to check that*

$\widehat{X}(S^*)$, as defined above, is the operator $f(S^*)$ defined through the usual H^∞ -functional calculus.

Example 4.2. In [7] Davidson and Pitts associate to every element of the algebra $\mathcal{L}_n = H^\infty(\mathbb{C}^n)$ a function on the open unit ball of \mathbb{C}^n . This is a special case of our analysis when $M = \mathbb{C}$, $E = \mathbb{C}^n$ and σ is a one dimensional representation of \mathbb{C} . In this case $\sigma(M)' = \mathbb{C}$ and $E^\sigma = \mathbb{C}^n$. Note, however, that our definition allows us to take σ to be the representation of \mathbb{C} on an arbitrary Hilbert space H . If we do so, then E^σ is isomorphic to $B(H)^{(n)}$, the n th column space over $B(H)$, and elements of \mathcal{L}_n define functions on the open unit ball of this space viewed as a correspondence over $B(H)$ with values in $B(H)$.

Note that, if $\xi_n \in E^{\otimes n}$ and $X = T_{\xi_n}$, we have, for $\eta \in AC(E^\sigma)$ and $h \in H$,

$$\widehat{T_{\xi_n}}(\eta^*)h = \eta_n^*(\xi_n \otimes h)$$

where, recall, $\eta_n = (I_{E^{\otimes(n-1)}} \otimes \eta) \cdots (I_E \otimes \eta)\eta : H \rightarrow E^{\otimes n} \otimes H$.

Example 4.3. Let G and $E = E(G)$ be as in Example 2.3. Let $\{e_v : v \in G^0\}$ be an orthonormal basis for a Hilbert space H and let σ be the diagonal representation of $\ell^\infty(G^0)$ on H (as at the end of Example 3.12). It follows from the computations in that example that, for the generators of $H^\infty(G)$, we get

$$(12) \quad \widehat{P_v}(\eta^*) = \theta_{v,v}, \quad v \in V$$

and

$$(13) \quad \widehat{S_e}(\eta^*) = \overline{\eta(e^{-1})} \theta_{r(e),s(e)}, \quad e \in \mathcal{Q}$$

where $\theta_{v,w}$ is the partial isometry operator on H that maps e_w to e_v and vanishes on $(e_w)^\perp$. For a general $X \in H^\infty(G)$, $\widehat{X}(\eta^*)$ is obtained by using the linearity, multiplicativity and w^* -continuity of the map $X \mapsto \widehat{X}(\eta^*)$.

In order to understand what functions can be obtained as \widehat{X} for some $X \in H^\infty(E)$, we first consider the following simple examples.

Example 4.4. Let $M = E = \mathbb{C}$ and σ be the one dimensional representation of \mathbb{C} . The Hardy algebra is the classical $H^\infty(\mathbb{D})$ and, for $X \in H^\infty(\mathbb{D})$, the function \widehat{X} is just X . Thus the functions we get are the functions on \mathbb{D} (which is $\mathbb{D}(E^\sigma)$ in this case) that are holomorphic and bounded. Recall that the Schur class \mathcal{S} is the set of all such functions S with $|S(z)| \leq 1$ (for $z \in \mathbb{D}$).

Example 4.5. Let $M = E = B(H)$ and σ be the identity representation of $B(H)$ on H . The Fock space $\mathcal{F}(E)$ is the direct sum of infinitely many copies of $B(H)$ and $\mathcal{F}(E) \otimes_\sigma H$ is (isomorphic to) $\ell^2 \otimes H$. The operator $T_I \otimes I_H$ is $S \otimes I_H$ where S is the unilateral shift on ℓ^2 and the operator $\varphi_\infty(A) \otimes I_H$ (for $A \in M = B(H)$) is $I_{\ell^2} \otimes A$. Since the induced representation $\sigma^{\mathcal{F}(E)}|_{H^\infty(E)}$ is completely isometric and a homeomorphism with respect to the ultra-weak topologies, we can identify $H^\infty(E)$ with $H^\infty(\mathbb{D}) \otimes B(H)$. The σ -dual E^σ in this case is \mathbb{C} and $\mathbb{D}(E^\sigma) = \mathbb{D}$. The transform $X \mapsto \hat{X}$ is just the expression of an element in $H^\infty(\mathbb{D}) \otimes B(H)$ as a bounded $B(H)$ -valued holomorphic function on \mathbb{D} . The set of all such functions S that satisfy $\|S(z)\| \leq 1$ (for all $z \in \mathbb{D}$) is known as the operator valued Schur class $\mathcal{S}(H)$.

Our objective in this section is to show that, in the general case, the functions we get as \hat{X} (for $X \in H^\infty(E)$ with $\|X\| \leq 1$) should be viewed as generalized Schur class functions.

For this, recall first, that the functions in the operator valued Schur class have several characterizations. The following is well known (see [5] for a more detailed exposition of the operator-valued Schur class and some of its generalizations).

Theorem 4.6. For an $B(H)$ -valued function S on \mathbb{D} the following conditions are equivalent.

- (1) $S \in \mathcal{S}(H)$; that is, S is a $B(H)$ -valued holomorphic function on \mathbb{D} with $\|S(z)\| \leq 1$ for all $z \in \mathbb{D}$.
- (2) There is a Hilbert space \mathcal{E} and a coisometric operator (called colligation)

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{E} \\ H \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{E} \\ H \end{pmatrix}$$

so that S can be realized as a linear fractional function

$$(14) \quad S(z) = D + zC(I_{\mathcal{E}} - zA)^{-1}B.$$

- (3) The function $K_S : \mathbb{D} \times \mathbb{D} \rightarrow B(H)$ given by

$$K_S(z, w) = \frac{I - S(z)S(w)^*}{1 - z\bar{w}}$$

is a positive kernel on $\mathbb{D} \times \mathbb{D}$ (with values in $B(H)$).

From the point of view of systems theory, a function S realized as in (14) is the transfer function of a certain linear system (defined using A, B, C and D).

For (3) above, recall that a function $K : \Omega \times \Omega \rightarrow B(H)$ is said to be a positive kernel if, for every $k \geq 1$ and every choice of $\omega_1, \dots, \omega_k$ in Ω , the matrix $(K(\omega_i, \omega_j))$ is positive (as an element of $M_k(B(H))$).

In order to discuss generalized Schur class operator functions we need to define a completely positive definite kernel. The definition below can be found in [6, Definition 3.2.2]. In fact, in that paper the authors show that this definition is equivalent to several other definitions and they prove an extension of Kolmogorov's representation theorem for these kernels.

Definition 4.7. *Let A, C be C^* -algebras and write $B(A, C)$ for the bounded (linear) maps from A to C . Let Ω be an arbitrary set. A function $K : \Omega \times \Omega \rightarrow B(A, C)$ is said to be a completely positive definite kernel if, for every $k \geq 1$ and every choice of $\omega_1, \dots, \omega_k$ in Ω , the map $\Psi_K : M_k(A) \rightarrow M_k(C)$ defined by $\Psi_K((a_{i,j})) = (K(\omega_i, \omega_j)(a_{i,j}))$ is completely positive.*

The following two theorems (Theorem 4.8 and Theorem 4.9), when compared with Theorem 4.6, show that we can indeed view the functions $\{\hat{X} : X \in H^\infty(E), \|X\| \leq 1\}$ as Schur class operator functions. The proofs can be found in [26].

Theorem 4.8. *Let E be a W^* -correspondence over M , σ a faithful normal representation of M on H and $Z : \mathbb{D}(E^\sigma)^* \rightarrow B(H)$. Then $Z = \hat{X}$ for some $X \in H^\infty(E)$ with $\|X\| \leq 1$ if and only if there is a Hilbert space \mathcal{E} , a normal representation τ of $\sigma(M)'$ on \mathcal{E} and a coisometric operator matrix*

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{E} \\ H \end{pmatrix} \rightarrow \begin{pmatrix} E^\sigma \otimes_\tau \mathcal{E} \\ H \end{pmatrix}$$

(with A, B, C, D that are $\sigma(M)'$ -module maps) so that Z can be realized as

$$Z(\eta^*) = D + C(I_{\mathcal{E}} - L_\eta^* A)^{-1} L_\eta^* B.$$

Here $L_\eta : \mathcal{E} \rightarrow E^\sigma \otimes_\tau \mathcal{E}$ is defined by $L_\eta h = \eta \otimes h$.

Theorem 4.9. *Let E be a W^* -correspondence over M , σ a faithful normal representation of M on H and $Z : \mathbb{D}(E^\sigma)^* \rightarrow B(H)$. Then $Z = \hat{X}$ for some $X \in H^\infty(E)$ with $\|X\| \leq 1$ if and only if the kernel $K_Z : \mathbb{D}(E^\sigma)^* \times \mathbb{D}(E^\sigma)^* \rightarrow B(\sigma(M)', B(H))$ is completely positive definite where*

$$K_Z(\eta^*, \zeta^*) = (id - Ad(Z(\eta^*), Z(\zeta^*))) \circ (id - \theta_{\eta, \zeta})^{-1}.$$

Here $Ad(Z(\eta^*), Z(\zeta^*))(a) = Z(\eta^*)aZ(\zeta^*)^*$ and $\theta_{\eta, \zeta}(a) = \langle \eta, a\zeta \rangle$ for $a \in \sigma(M)'$.

In [26] we used the condition appearing in Theorem 4.9 to define a Schur class operator function.

Definition 4.10. *Let Ω be a subset of $\mathbb{D}(E^\sigma)$ and let $\Omega^* = \{\omega^* \mid \omega \in \Omega\}$. A function $Z : \Omega^* \rightarrow B(H)$ will be called a Schur class operator function (with values in $B(H)$) if the kernel $K_Z : \Omega^* \times \Omega^* \rightarrow B(\sigma(M)', B(H))$, defined by*

$$K_Z(\eta^*, \zeta^*) = (id - Ad(Z(\eta^*), Z(\zeta^*))) \circ (id - \theta_{\eta, \zeta})^{-1}, \quad \eta, \zeta \in \Omega$$

is completely positive definite.

Thus, we see that the functions of the form \widehat{X} are precisely the Schur class operator functions on the open unit ball of E^σ .

One direction of Theorem 4.9 follows from a Nevanlinna-Pick type interpolation theorem that we proved in [25, Theorem 5.3]. Before we state it, recall the classical Nevanlinna-Pick theorem.

Theorem 4.11. *Given z_1, \dots, z_m in \mathbb{D} and w_1, \dots, w_m in \mathbb{C} , one can find a function $f \in H^\infty(\mathbb{D})$ with $\|f\| \leq 1$ and $f(z_i) = w_i$ for all i if and only if the $m \times m$ matrix*

$$\left(\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}} \right)$$

is positive.

Our generalization is Theorem 4.12. This result captures numerous theorems in the literature that go under the name of generalized Nevanlinna-Pick theorems. In particular, of course, it gives the classical Nevanlinna-Pick theorem (when $M = E = \mathbb{C}$). In the setting when $M = \mathbb{C}$ and $E = \mathbb{C}^n$, it gives versions due to Popescu in [31], Arias and Popescu in [2] and Davidson and Pitts in [8].

Theorem 4.12. ([25, Theorem 5.3]) *Let E be a W^* -correspondence over a von Neumann algebra M and let $\sigma : M \rightarrow B(H)$ be a faithful normal representation of M on a Hilbert space H . Fix k points η_1, \dots, η_k in the disk $\mathbb{D}(E^\sigma)$ and choose $2k$ operators $B_1, \dots, B_k, C_1, \dots, C_k$ in $B(H)$. Then there exists an X in $H^\infty(E)$ such that $\|X\| \leq 1$ and*

$$B_i \widehat{X}(\eta_i^*) = C_i$$

for $i = 1, 2, \dots, k$, if and only if the map from $M_k(\sigma(M)')$ to $M_k(B(H))$ defined by the $k \times k$ matrix

$$(15) \quad ((Ad(B_i, B_j) - Ad(C_i, C_j)) \circ (id - \theta_{\eta_i, \eta_j})^{-1})$$

is completely positive.

The proof was inspired by Sarason's approach that is based on the commutant lifting theorem [35] and the extensions of it to the multi-analytic setting of [31]. The organization of our proof follows the presentation of [34] and one of the key ingredients is our commutant lifting theorem (Theorem 3.11). Another important ingredient in the proof is the following theorem that identifies the commutant of an induced representation and points out another role that the σ -dual E^σ plays in studying the Hardy algebra $H^\infty(E)$.

In order to state the theorem, recall that if E is a W^* -correspondence over the von Neumann algebra M and σ is a faithful normal representation of M on H , we can represent $H^\infty(E)$ on $\mathcal{F}(E) \otimes_\sigma H$ using the induced representation. Write ρ for this representation so that $\rho(X) = \sigma^{\mathcal{F}(E)}(X) = X \otimes I_H$ for $X \in H^\infty(E)$. Note that this representation is completely isometric isomorphism and a homeomorphism with respect to the ultra-weak topologies. Similarly, we have an induced representation ρ' of $H^\infty(E^\sigma)$ on $\mathcal{F}(E^\sigma) \otimes_\iota H$ where ι is the identity representation of $\sigma(M)'$ on H .

Theorem 4.13. ([25, Theorem 3.9]) *Let E be a W^* -correspondence over the von Neumann algebra M , let σ be a faithful normal representation of M on H and let ρ and ρ' be as above. Then the commutant of $\rho(H^\infty(E))$ is unitarily isomorphic to $\rho'(H^\infty(E^\sigma))$.*

Consequently (using duality arguments), $(\rho(H^\infty(E)))'' = \rho(H^\infty(E))$.

Using the Nevanlinna-Pick theorem, we get another interesting result that fits with the "noncommutative function theory" point of view. This is the generalization of Schwartz's lemma (see [25, Theorem 5.6]). It asserts, among several things, that for $X \in H^\infty(E)$, if \widehat{X} vanishes at the origin and if $\|X\| \leq 1$, then

$$\widehat{X}(\eta^*)\widehat{X}(\eta^*)^* \leq \langle \eta, \eta \rangle, \quad \eta \in \mathbb{D}(E^\sigma)$$

where, recall, $\langle \cdot, \cdot \rangle$ is the $\sigma(M)'$ -valued inner product on E^σ defined above. (See Proposition 3.5).

The generalized Nevanlinna-Pick theorem (Theorem 4.12), as stated above, interpolates the values of the function at points in the open unit ball $\mathbb{D}(E^\sigma)^*$. But we now know that functions of the form \widehat{X} are defined on the, possibly larger, set $AC(E^\sigma)^*$. We wish to present a similar interpolation theorem where the points η_1, \dots, η_k are from $AC(E^\sigma)$.

The first problem that one encounters in trying to do this is that the maps $(id - \theta_{\eta_i, \eta_j})$, appearing in the theorem, are not necessarily invertible (as we may have $\|\eta_i\| = 1$). The theorem, therefore, will have

to be stated differently. In order to deal with points on the boundary we use the following simple observation.

Simple observation : If Φ, Ψ are positive maps such that $id - \Phi$ is invertible then the map $(id - \Psi) \circ (id - \Phi)^{-1}$ is positive if and only if :

$$\{a \geq 0 : \Phi(a) \leq a\} \subseteq \{a \geq 0 : \Psi(a) \leq a\}.$$

The last statement makes sense even if $id - \Phi$ is not invertible. It is related to the Lyapunov preorder studied in matrix theory. This was pointed out to us by Nir Cohen.

Using this observation and the characterizations of the points in $AC(E^\sigma)$ (Theorem 3.17) we get the following.

Theorem 4.14. *Let E be a W^* -correspondence over the von Neumann algebra M and let σ be a faithful normal representation of M on H . Given $\eta_1, \eta_2, \dots, \eta_k \in AC(E^\sigma)$ and $D_1, D_2, \dots, D_k \in B(H)$, the following conditions are equivalent.*

- (1) *There is an element $X \in H^\infty(E)$ such that $\|X\| \leq 1$ and such that*

$$\widehat{X}(\eta_i^*) = D_i,$$

$i = 1, 2, \dots, k$.

- (2) *For each $m \geq 1$, $i : \{1, \dots, m\} \rightarrow \{1, \dots, k\}$ and C_1, C_2, \dots, C_m with $C_j \in \mathcal{I}(S_0, \eta_{i(j)}^*)$, we have*

$$(D_{i(l)} C_l C_j^* D_{i(j)}^*)_{l,j} \leq (C_l C_j^*)_{l,j}.$$

In the case where $M = E = \mathbb{C}$, the above theorem gives an answer to the question: Given k contractions T_1, T_2, \dots, T_k in $B(H)$ that have H^∞ -functional calculus and k operators D_1, D_2, \dots, D_k in $B(H)$, when can we find a function $h \in H^\infty(\mathbb{D})$ such that $D_i = h(T_i)$ for all $1 \leq i \leq k$?

5. THE KERNEL OF THE TRANSFORM $X \mapsto \widehat{X}$ AND QUOTIENT ALGEBRAS

In the last section we discussed the map $X \mapsto \widehat{X}$ that maps every element X of $H^\infty(E)$ to a function \widehat{X} , defined on $\mathbb{D}(E^\sigma)^*$ or $AC(E^\sigma)^*$, that was seen to be a Schur class operator function. We noted there that this transform depends on σ and it may have a kernel. This was observed already by Davidson and Pitts in [7]. They showed that, when $M = \mathbb{C}$, $E = \mathbb{C}^n$ and σ is the one-dimensional representation of \mathbb{C} (see Example 4.2), the kernel of this map is the commutator ideal of $H^\infty(\mathbb{C}^n)$.

In general, we write $K(\sigma)$ for this kernel. Thus

$$(16) \quad K(\sigma) = \cap \{Ker(\eta^* \times \sigma) : \eta \in \mathbb{D}(E^\sigma)\}.$$

The following lemma was proved in [26, Lemmas 3.7 and 4.17].

Lemma 5.1. *Let σ be a normal, faithful, representation of M on a Hilbert space H and let $K(\sigma)$ be defined by (16). Then*

- (1) $K(\sigma)$ is an ultra-weakly closed ideal of $H^\infty(E)$.
- (2) $K(\sigma) \subseteq \{X \in H^\infty(E) : \Phi_0(X) = \Phi_1(X) = 0\}$ (where Φ_k were defined in (3)).
- (3) $K(\sigma)$ is invariant under the action of the gauge group and, thus, under the maps Φ_k , $k \geq 0$.
- (4) If σ is of infinite multiplicity, then $K(\sigma) = \{0\}$.

It follows from statement (4) that we can always choose σ such that this transform is injective.

For a general representation σ , we can view this transform as a map on the quotient algebra $H^\infty(E)/K(\sigma)$ (into the Schur class operator functions). Note, however, that, as shown by Arveson [3], this map is not isometric when the Schur class functions are viewed with the supremum norm. The following theorem identifies quotient algebras of $H^\infty(E)$ with algebras that are obtained by compressing $H^\infty(E)$ into a coinvariant subspace of $\mathcal{F}(E)$. For the case $M = \mathbb{C}$ and $E = \mathbb{C}^n$, this was proved by Davidson and Pitts in [8]. The general result was proved independently by J. Meyer ([18]) and by M. Gurevich ([11]). A norm-closed version was proved by A. Viselter ([39]).

Theorem 5.2. *Let $J \subseteq H^\infty(E)$ be an ultra-weakly closed two-sided ideal in $H^\infty(E)$. Write \mathcal{M} for the closed submodule $\overline{J\mathcal{F}(E)}$ and $P \in \mathcal{L}(\mathcal{F}(E))$ for the projection onto \mathcal{M}^\perp . Then the map $X \in H^\infty(E) \mapsto PXP$ induces a complete isometric isomorphism mapping the quotient algebra $H^\infty(E)/J$ onto $PH^\infty(E)P$.*

When J is invariant for the action of the gauge group on $H^\infty(E)$, as is the case for the ideal $K(\sigma)$, the algebra $PH^\infty(E)P$ (for P as in the theorem) is the Hardy algebra of a *subproduct system*. Subproduct systems were defined and studied in [36]. Theorem 5.2 shows that they can be useful in studying the quotient algebras $H^\infty(E)/K(\sigma)$. We shall not discuss it further here and the interested reader is referred to [36] and [39].

6. VARYING σ

In most of the discussion above we fixed a normal representation σ and, for $X \in H^\infty(E)$, considered the function \hat{X} defined on $\mathbb{D}(E^\sigma)^*$ or on $AC(E^\sigma)^*$. Now we let σ vary. We fix M and E and write Σ for the set of all normal representations σ of M on some Hilbert space H_σ .

For every $X \in H^\infty(E)$ and every $\sigma \in \Sigma$, we write \widehat{X}_σ for the (Schur class operator) function associated to X on $AC(E^\sigma)^*$. We get a family of operator valued functions $\{\widehat{X}_\sigma : \sigma \in \Sigma\}$.

In this section we discuss the relationships among the functions in this family. Our discussion was inspired by several sources. First, there is the pioneering paper by Joe Taylor [38]. This paper seems to have generated very little interest until relatively recently. But on close reading, it is clear that it was extraordinarily prescient. It had a big impact on Dan Voiculescu's "free analysis" questions [40] and most recently it helped to shape the foundations of noncommutative function theory being developed by Dmitry Kalyuzhnyi-Verbovetzkiĭ and Victor Vinnikov and the applications of it to linear matrix inequalities and real algebraic geometry in the work of Bill Helton, Igor Klepp and Scott McCullough and their collaborators (see, e.g., [12]). And of course, it has been a direct source of inspiration for our work.

Each function \widehat{X}_σ is defined on its own domain, $AC(E^\sigma)^*$, and these domains vary with σ . One can therefore view $\{\widehat{X}_\sigma\}_{\sigma \in \Sigma}$ as a single function $\widehat{\mathbf{X}}$ from $\mathcal{AC}(E) := \coprod_{\sigma \in \Sigma} AC(E^\sigma)^*$ to $\mathcal{B} := \coprod_{\sigma \in \Sigma} B(H_\sigma)$ with the property that $\widehat{\mathbf{X}}$ maps $AC(E^\sigma)^*$ to $B(H_\sigma)$. In fact, it will be convenient to view \mathcal{B} as a bundle over $\mathcal{AC}(E)$ with the property that the total space of $\mathcal{B}|_{AC(E^\sigma)^*}$ is $AC(E^\sigma)^* \times B(H_\sigma)$. When this is done, we follow the customary practice of identifying a section of a trivial bundle over a space with a function on the space with values in the fibre. (In this case the trivial bundle is $AC(E^\sigma)^* \times B(H_\sigma)$ over $AC(E^\sigma)^*$.) Then we can say, simply, that $\widehat{\mathbf{X}}$ is a *section* of this bundle and adopt the following terminology.

Definition 6.1. *The section $\widehat{\mathbf{X}}$ of the bundle $\mathcal{B} = \coprod_{\sigma \in \Sigma} AC(E^\sigma)^* \times B(H_\sigma)$ over $\mathcal{AC}(E) = \coprod_{\sigma \in \Sigma} AC(E^\sigma)^*$ associated with the element $X \in H^\infty(E)$ is called the complete Schur class section determined by X .*

Remark 6.2. *It is a consequence of [26, Lemma 3.8] and the fact that we incorporate all the normal representations of M that the map $X \rightarrow \widehat{\mathbf{X}}$ is injective. It is very much of interest to understand how to adjust matters when one restricts attention to some subset of Σ .*

A natural question is, "How does one recognize such a section?". The answer begins with the structure of the families of sets $\mathcal{AC}(E) = \{AC(E^\sigma) : \sigma \in \Sigma\}$ and $\mathcal{D}(E) := \{\mathbb{D}(E^\sigma) : \sigma \in \Sigma\}$, which is abstracted by the following definition.

Definition 6.3. *A family $\mathcal{A} = \{\mathcal{A}(\sigma) : \sigma \in \Sigma\}$ is said to be a fully matricial E-set if*

- (i) for each σ , $\mathcal{A}(\sigma) \subseteq E^\sigma$,
- (ii) it is closed with respect to taking direct sums; that is, $\mathcal{A}(\sigma) \oplus \mathcal{A}(\tau) \subseteq \mathcal{A}(\sigma \oplus \tau)$ and
- (iii) it is closed with respect to unitary similarity; that is, if $\eta \in \mathcal{A}(\sigma)$ and $u \in \sigma(M)'$ is a unitary then $u \cdot \eta \cdot u^* \in \mathcal{A}(\sigma)$.

Note that the product $u \cdot \eta \cdot u^*$ is the bimodule product on E^σ .

Our notion is very similar to one defined by Taylor in [38] and to one defined by Voiculescu in [40]. In these papers, the general linear group is used instead of the unitary group. For our purposes here, however, it is more convenient to work with the unitary group, as Helton, Klepp and McCullough did in [12]. It is evident that just as with $\mathcal{AC}(E)$, we may view \mathcal{B} as a bundle over any fully matricial E -set and study sections of this bundle. The following definition singles out a special property that a section may or may not have.

First, we need to extend the definition of the spaces $\mathcal{I}(S_0, \eta^*)$ that we have met before: For *any* two $\sigma, \tau \in \Sigma$, and $\eta \in E^\sigma$ and $\zeta \in E^\tau$, we write $\mathcal{I}(\eta^*, \zeta^*)$ for the collection of all $C : H_\sigma \rightarrow H_\tau$ such that $C\sigma(\cdot) = \tau(\cdot)C$ and such that $C\eta^* = \zeta^*(I_E \otimes C)$, as maps from $E \otimes_\sigma H_\sigma$ to H_τ . We call an element of $\mathcal{I}(\eta^*, \zeta^*)$ an intertwiner of η^* and ζ^* . It is easy to see that if η and ζ both have norm at most one, then $\mathcal{I}(\eta^*, \zeta^*)$ is simply the collection of operators that intertwine $\sigma \times \eta^*$ and $\tau \times \zeta^*$. Note that an intertwiner $C \in \mathcal{I}(\eta^*, \zeta^*)$ will also satisfies $C\eta_k^* = \zeta_k^*(I_{E^{\otimes k}} \otimes C)$ for all $k \geq 1$ where, recall, η_k^* and ζ_k^* are the generalized powers of η^* and ζ^* discussed above.

Definition 6.4. Let $\mathcal{A} = \{\mathcal{A}(\sigma) : \sigma \in \Sigma\}$ be a fully matricial E -set and form the bundle $\mathcal{B} = \coprod_{\sigma \in \Sigma} \mathcal{A}(\sigma) \times B(H_\sigma)$. We say that a section \mathbf{f} of \mathcal{B} preserves intertwiners in case $Cf_\sigma(\eta) = f_\tau(\zeta)C$ for all $\sigma, \tau \in \Sigma$, $(\eta, \zeta) \in \mathcal{A}(\sigma) \times \mathcal{A}(\tau)$, and intertwiners $C \in \mathcal{I}(\eta^*, \zeta^*)$.

With all the pieces before us, we may formulate our “recognition” theorem as follows.

Theorem 6.5. A section \mathbf{f} of the bundle \mathcal{B} over $\mathcal{AC}(E)$ is a complete Schur section if and only if \mathbf{f} preserves intertwiners.

We won’t go into the details of the proof here, but we do want to point out that a special role is played by the fact that the functions are defined on \mathcal{AC} and not just on the family of open sets $\mathcal{D}(E) = \coprod_{\sigma \in \Sigma} \mathbb{D}(E^\sigma)^*$. A key role is played by our Theorem 3.23; the interaction between “absolute continuity” and the special nature of elements in $H^\infty(E)$ is what underlies the proof.

Now a section $\widehat{\mathbf{X}}$, $X \in H^\infty(E)$, may be restricted to $\mathcal{D}(E)$ and when this is done, the resulting section is analytic as a Banach-space-valued

section, i.e., $\widehat{X_\sigma}$ is a $B(H_\sigma)$ -valued analytic function on $\mathbb{D}(E^\sigma)^*$. But there are many sections of $\mathcal{B}|_{\mathcal{D}(E)}$ with this property that don't come from elements of $H^\infty(E)$. Here is a very simple example.

Example 6.6. *Let $M = E = \mathbb{C}$. In this case $H^\infty(E)$ is the classical $H^\infty(\mathbb{D})$. The representations in Σ are just the obvious representations. We let σ be the identity representation of \mathbb{C} on \mathbb{C} . Then every representation of \mathbb{C} is a multiple of σ , $n\sigma$, which acts on \mathbb{C}^n . We treat \mathbb{C}^∞ as $\ell^2(\mathbb{N})$. Then $E^{n\sigma} = E^{n\sigma*} = B(\mathbb{C}^n)$ and $\mathbb{D}(E^{n\sigma})^* = \{A \in B(\mathbb{C}^n) \mid \|A\| < 1\}$. We set $f_{n\sigma}(A) = (I - A)^{-1}$, for $A \in \mathbb{D}(E^{n\sigma})^*$.*

If $A \in B(\mathbb{C}^n)$, $B \in B(\mathbb{C}^m)$ both have norm less than 1 and if $C : \mathbb{C}^m \rightarrow \mathbb{C}^n$ intertwines them, that is, if $AC = CB$, then C also intertwines $(I - A)^{-1}$ and $(I - B)^{-1}$. Thus $\mathbf{f} = \{f_{n\sigma}\}$ is a section of $\mathcal{B}|_{\mathcal{D}(E)}$ which certainly deserves to be called analytic. After all, it comes from the function h , where $h(z) = \sum_{n \geq 0} z^n$. However, h is not in $H^\infty(\mathbb{D})$.

Note that in this example, we have identified $H^\infty(\mathbb{D})$, which is a space of complex valued functions on the (classical) unit disc $\mathbb{D} = \mathbb{D}(E^\sigma)^*$, with $H^\infty(\mathbb{C})$, which really is a space of sequences, viz., the space of the sequences of Taylor coefficients of the functions in $H^\infty(\mathbb{D})$. In the case of h , of course, the sequence is $(1, 1, 1, \dots)$, which is not in $H^\infty(\mathbb{C})$. In general, recall, every element $X \in H^\infty(E)$ has a series expansion (4), so it is natural to wonder if it is possible to manipulate arbitrary series of tensors. It is possible, and to help clarify how, we introduce the following definition.

Definition 6.7. *Let E be a W^* -correspondence over the W^* -algebra M .*

- (1) *A (formal) series of tensors (over E) is simply a sequence $\theta = \{\theta_k\}_{k \geq 0}$, where $\theta_k \in E^{\otimes k}$. However, we shall usually write $\theta \sim \sum_{k \geq 0} \theta_k$ in anticipation of function-theoretic considerations to come.*
- (2) *If $\theta \sim \sum_{k \geq 0} \theta_k$ is a series of tensors over E , then we define $R(\theta)$ to be*

$$(\overline{\lim}_k \|\theta_k\|^{1/k})^{-1},$$

and we refer to $R(\theta)$ as the radius of convergence of θ .

Evidently, $R(\theta)$ is a non-negative number or $+\infty$. The formula for $R(\theta)$ suggests that $\sum_{k \geq 0} \theta_k$ converges in some sense. And of course it does, as the following theorem shows. It is a generalization of the well known Cauchy-Hadamard theorem from elementary complex analysis. It plays a prominent role in Popescu's study of free analyticity, also.

Theorem 6.8. *Suppose $\theta \sim \sum_{k \geq 0} \theta_k$ is a series of tensors coming from E , and let $R = R(\theta)$ be its radius of convergence.*

- (1) *Given $\sigma \in \Sigma$ and $\eta^* \in R\mathbb{D}(E^\sigma)^* := \{R\zeta^* \mid \zeta^* \in \mathbb{D}(E^\sigma)^*\}$, the series $\sum_k \|\eta_k^* L_{\theta_k}\|$ converges, where, recall, η_k^* denotes the k^{th} generalized power of η^* and where L_{θ_k} is the map from H to $E^{\otimes k} \otimes_\sigma H$ defined by $L_{\theta_k} h = \theta_k \otimes h$. If $0 < \rho < R$, the convergence is uniform on $\rho\mathbb{D}(E^\sigma)$.*
- (2) *If $R < R' < \infty$, there exists a $\sigma \in \Sigma$ and an $\eta \in E^\sigma$ with $\|\eta\| = R'$ such that $\sum_k \|\eta_k^* L_{\theta_k}\| = \infty$.*

Thus, a series θ defines a function on each of the discs $R\mathbb{D}(E^\sigma)^*$ with values in $B(H_\sigma)$. We denote this function by $\widehat{\theta}_\sigma$. Its value at an η^* is given by the formula,

$$\widehat{\theta}_\sigma(\eta^*) = \sum_{k \geq 0} \eta_k^* L_{\theta_k}.$$

By Theorem 6.8, this series is a series of operators $B(H_\sigma)$ that converges in norm. The family $\{\widehat{\theta}_\sigma\}_{\sigma \in \Sigma}$ forms the section $\widehat{\theta}$ of the bundle $\mathcal{B} := \coprod_{\sigma \in \Sigma} R\mathbb{D}(E^\sigma)^* \times B(H_\sigma)$ over $R\mathcal{D}(E) := \coprod_{\sigma \in \Sigma} R\mathbb{D}(E^\sigma)^*$.

Definition 6.9. *Let $\theta \sim \sum_{k \geq 0} \theta_k$ be a series of tensors over E and let R be its radius of convergence. The section $\widehat{\theta}$ of the bundle $\mathcal{B} := \coprod_{\sigma \in \Sigma} R\mathbb{D}(E^\sigma)^* \times B(H_\sigma)$ over $R\mathcal{D}(E) := \coprod_{\sigma \in \Sigma} R\mathbb{D}(E^\sigma)^*$ determined by the family $\{\widehat{\theta}_\sigma\}_{\sigma \in \Sigma}$ is called the free analytic section determined by θ .*

We may now characterize the free analytic sections in much the same fashion as we characterized complete Schur sections.

Theorem 6.10. *A section $\mathbf{f} = \{f_\sigma\}_{\sigma \in \Sigma}$ of the bundle*

$$\mathcal{B} = \coprod_{\sigma \in \Sigma} R\mathbb{D}(E^\sigma)^* \times B(H_\sigma) \text{ over } R\mathcal{D}(E)$$

is the free analytic section determined by a series of tensors with radius of convergence at least R if and only if \mathbf{f} preserves intertwiners.

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